

# A statistical approach to rod filtration

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**Abstract.** We investigate the efficacy of a square-grid mesh at filtering rod-like granular particles from solution. The volume fraction  $\phi$  is kept low so particles cannot cooperatively jam. Our work extends the Buffon-Laplace Needle problem, one of the earliest statistical calculations, to three dimensions, accounts for finite widths, and finds the probability for particles to make contact with a 2D sieve-like mesh. This is a necessary but not sufficient condition for filtration. We then assume a frictionless interaction with the mesh to account for particle rotations that occur after contact, and find that the intersection probability neither vanishes for very short particles nor goes to unity for very long particles. We argue that our assumptions produce an upper bound on the filtration probability and confirm this with comparison with a simple experiment in which cylindrical particles are filtered with a square-hole mesh.

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## 1. Introduction

The ability to separate granular materials by size is of fundamental importance to industrial handling of these materials. Segregating large round particles from small requires nothing more than a mesh with hole size intermediate between the two particle diameters. While some clogging[1] inevitably occurs, this can usually be addressed by other means (e.g. vibration, [2]).

For rod-like granular materials, however, the problem is significantly more complicated. Rods have two length scales — diameter and length — that can differ by over an order of magnitude. Separating the components of a binary mixture of rods with different lengths but similar diameters is a nontrivial task. Holes must be larger than the rods' diameters, but once this condition is satisfied, even long rods can pass through if oriented perpendicular to the plane of the mesh. An efficient way to separate rods is not known. The situation is further complicated by the complex network of voids [3] and contacts [4] in the pile that can result in solid-like structures [5].

Recently, statistical or probabilistic approaches have been used to explain a variety of phenomena involving jamming of round granular particles. To et al.[6, 7, 8] used a random-walk model to explain the different configurations by which two-dimensional disks jam at a hopper entrance. Zuriguel [9, 10] measured the number of particles that exited a conical hopper before a jam occurred and found an exponentially decaying probability distribution. He interpreted this as resulting from an uncorrelated particle exit probability  $p$ ; the probability for  $n$  particles to exit is therefore  $P(n) \propto p^n = \exp[n \ln p]$ .

Recently, Roussel et al. [1] investigated the jamming of round particles passing through holes in a square mesh. A jam occurred when multiple particles collectively formed an arch spanned the mesh hole. This was explained as the product of two statistically independent events. First, a single particle encounters the hole boundary with a probability dependent solely on the geometry of the particle and mesh. The particle that first comes in contact with the mesh, they speculated, slows down as it moves around the wire, allowing other particles time to catch up and form the arch. The probability for this to occur depends on the flow velocity, particle volume fraction, and number of particles needed to span the hole.

In this paper we consider the low-volume fraction filtration of rods. The low concentration removes the possibility that particles can cooperatively jam, reducing the problem to the geometric question of how they intersect with the mesh boundary. We find that this probability is not trivial, not vanishing even in the limit of large mesh size nor reaching unity in the limit of large particle size. This is fundamentally different from the situation with spheres; a mesh with holes larger than the particle diameter will not capture any spheres if the concentration is low.

The stability of a rod in contact with a mesh can be defined as one satisfying torque balance, requiring two contacts that bracket the particle center of mass. Assuming a frictionless interaction between mesh and particle allows us to extend the analysis to

include particle dynamics after contact is made and compare with a simple experiment.

In Section 2 we review the Buffon-Laplace Needle problem, a two-dimensional statistical approach upon which our model builds, and extend the calculation to account for finite particle widths. In Section 3 we consider rod distributions in three-dimensions, showing that the jamming probability never vanishes. In Section 4 we consider a simple approach to what happens after contact with the mesh and, in Section 5, compare these results with experiment.

## 2. The Buffon-Laplace Needle Problem

In 1770 Buffon[11, 12] considered the probability that an infinitesimally thin needle of length  $l$  dropped on a surface marked by a series of infinite parallel lines spaced  $d$  apart would intersect one of the lines. (Henceforth all lengths are in units of the line spacing  $d$ .) The solution is found by considering only two lines, one at  $y = 0$  and the other at  $y = 1$ , and the needle's center of mass may lie at any height between 0 and 1. The needle orientation takes any angle  $\theta \in \pm\pi/2$  with uniform probability. An intersection is avoided when the center of mass position is within  $[l |\sin \theta| / 2, 1 - l |\sin \theta| / 2]$ . Averaging this range over all orientations (and subtracting from 1) results in the intersection probability

$$P_{\text{int}}(l < 1) = 1 - \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} (1 - l |\sin \theta|) d\theta = \frac{2l}{\pi}. \quad (1)$$

When the needle length equals the line spacing ( $l = 1$ ) the intersection probability is  $2/\pi = 0.637$ , indicating non-intersecting orientations about  $\theta = 0$ . The solution for  $l > 1$  is found by restricting the range in  $\theta$  to  $\theta \in \pm \arcsin(1/l)$ , with intersection probability

$$P_{\text{int}}(l > 1) = 1 - \frac{1}{\pi} \int_{-\arcsin(1/l)}^{\arcsin(1/l)} (1 - l |\sin \theta|) d\theta \quad (2)$$

$$= \frac{2}{\pi} \left( l - \sqrt{l^2 - 1} + \sec^{-1} l \right). \quad (3)$$

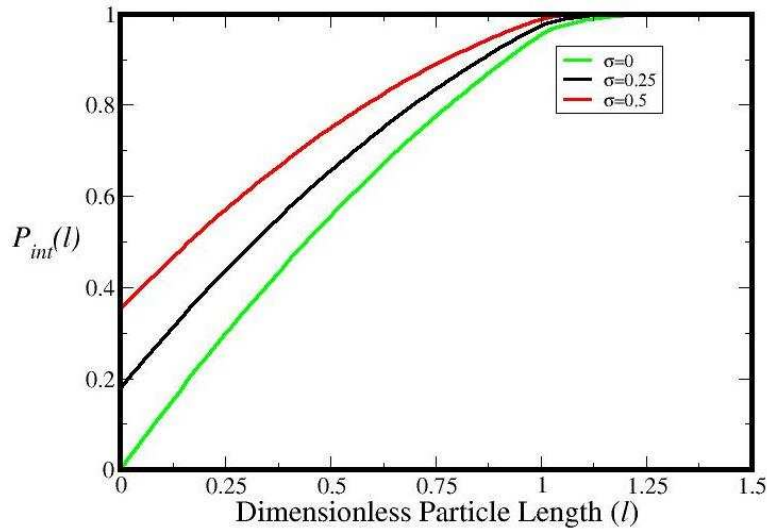
This expression has the correct large  $l$  asymptote of  $P(l \rightarrow \infty) = 1$ . This analysis has been used to find experimental values for  $\pi$  by dropping needles and counting the intersections, with at least one suspected case of fraud [13].

In 1812 Laplace [14, 15] generalized the analysis to a needle dropped on a rectangular mesh of arbitrary length and width (we only consider the square mesh of sidelength 1). The center of mass samples an allowable area  $A(\theta) = (1 - l \cos \theta)(1 - l |\sin \theta|)$ , with the result that

$$P_{\text{int}}(l < 1) = 1 - \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} A(\theta) d\theta = \frac{l(4 - l)}{\pi}. \quad (4)$$

It does not appear that Laplace extended his analysis to longer lengths, although it is clear that not until  $l = \sqrt{2}$  is an intersection guaranteed. Changing the limits in the integral in Eq. 4 to  $\pm \arcsin(1/l)$  (as in 2), results in an intersection probability of

$$P_{\text{int}}(l > 1) = 1 - \frac{1}{\pi} \int_{-\arcsin(1/l)}^{\arcsin(1/l)} A(\theta) d\theta \quad (5)$$



**Figure 1.** Probability of spherocylinders with dimensionless diameters  $\sigma = 0$  (bottom),  $\sigma = 0.25$  (middle), and  $\sigma = 0.5$  (top) placed at random positions and orientation on a 2-d grid to intersect the boundary of a square hole of sidelength 1. As expected, thicker particles have a greater chance of intersection.

$$= 1 - \frac{1}{\pi} \left[ \arcsin(1/l) - \arccos(1/l) + 2\sqrt{l^2 - 1} - l^2/2 - 1 \right] \quad (6)$$

This expression has the correct limit that  $P_{\text{int}}(l = \sqrt{2}) = 1$ .

To make the Buffon-Laplace results applicable to actual particles, we first generalize the analysis to spherocylinders [16], a cylinder (dimensionless diameter  $\sigma$ ) with hemispherical endcaps. All points on the surface of the spherocylinder are a distance  $\sigma/2$  from the central line segment. The total length of the spherocylinder is  $l + \sigma$  and its aspect ratio  $1 + l/\sigma$ , with  $l = 0$  corresponding to a sphere. The finite particle width reduces the total area available to the center of mass to

$$A(\theta, \sigma) = (1 - \sigma - l \cos \theta)(1 - \sigma - l |\sin \theta|) \quad (7)$$

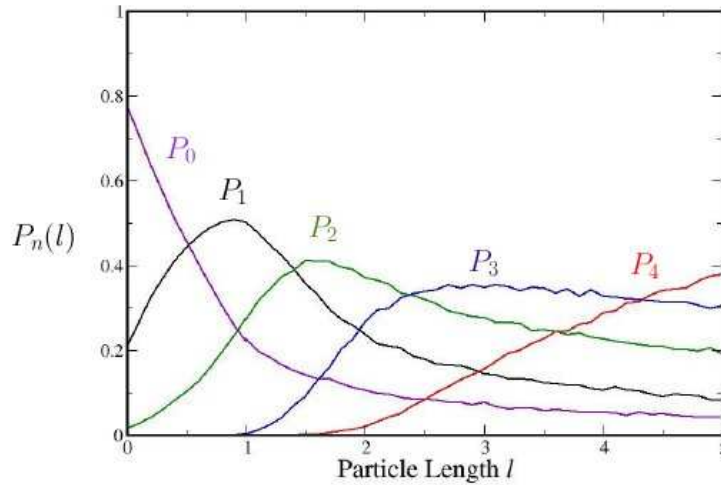
and the intersection probability is found to be

$$P_{\text{int}}(l < 1) = \frac{4l - l^2}{\pi} + \frac{\sigma}{\pi} [(2\pi - 4) - \sigma(\pi - 3) - 2l]. \quad (8)$$

$P_{\text{int}}(l > 1)$  can also be found analytically, although it is a significantly more complicated expression. The intersection probability is plotted in Fig. 1; as expected, spherocylinders with larger diameters have a larger probability of intersecting the boundary. The probability  $P(l)$  does not reach one until  $l \approx \sqrt{2} - \sigma$ , corresponding to the particle lying along the diagonal of the square-shaped hole.

### 3. Three Dimensions

We now extend the Buffon-Laplace analysis to particles assuming arbitrary orientation in three-dimensions and consider the probability that the length projected onto the



**Figure 2.** Probability  $P_n(l)$  of spherocylinders with dimensionless diameter  $\sigma = 0.125$  and cylinder length  $l$  isotropically oriented in three dimensions to intersect *exactly*  $n$  gridlines of the mesh.

mesh intersects a hole boundary. Assuming an  $xyz$  coordinate system with the mesh lying in the  $xy$ -plane, rods assume, with equal probability, any angle  $\phi$  with respect to the vertical ( $\hat{z}$ ) direction. The probability of intersecting with the mesh depends on the projected length  $l \sin \phi$ , and even the longest rod has a small percentage of nearly vertical non-intersecting orientations. This simple geometric argument explains the fundamental difficulty of filtering rod-like particles. The area available to the particle center of mass (without intersection) is now a function of the projected length, and hence  $\phi$ :

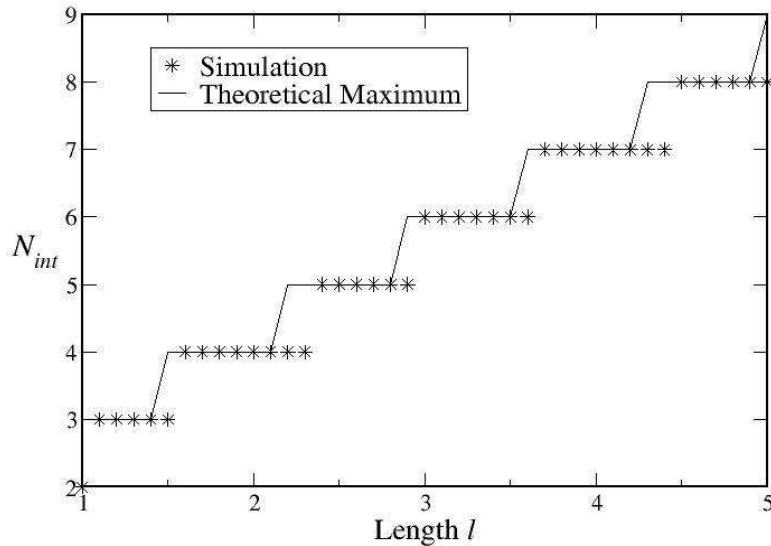
$$A(l, \theta, \phi) = (d - \sigma - l \sin \phi \cos \theta)(d - \sigma - l \sin \phi |\sin \theta|).$$

This must be integrated over all orientations in both  $\theta$  and  $\phi$  to obtain the probability for intersection:

$$P_{\text{int}}(l) = 1 - \int_0^{\pi/2} d\phi \int_{\theta_{\min}}^{\theta_{\max}} A(l, \theta, \phi) d\theta \quad (9)$$

This averaging can either be done numerically or by Monte-Carlo simulation; we have confirmed that the two approaches produce the same result. Results are presented from the Monte-Carlo simulations, since that allows us to later extend our discussion to include particle rotations after an intersection occurs.

Figure 2 shows the probability  $P_n(l)$  that the projected length of a rod (length  $l$ ) intersects zero ( $P_0$ ), one ( $P_1$ ), two ( $P_2$ ), three ( $P_3$ ), or four ( $P_4$ ) lines of the mesh. There are several features in Fig. 2 worth noting. The probability that a rod does not contact a boundary  $P_0$  is not 0 even for long rods. Second, if we assume that contact with two gridlines is a necessary, if not sufficient, condition for a rod to be caught, then Fig. 2 shows that this probability is non-zero even when the  $l < 1$ , when the rod length is smaller than the hole sidelength. This occurs when a rod spans the corner of the square mesh. These two features are particularly troubling when considering the



**Figure 3.** Maximum number of gridlines crossed as a function of rod length  $l$ . (\*) data from simulations; — the theoretically maximum number of intersections  $N_{int}^{max} = 2 + \sqrt{2}l$ . Because the theoretical calculation assumes an intersection exactly at the origin, the jump to higher  $N_{int}$  occurs at a slightly lower length than in simulations.

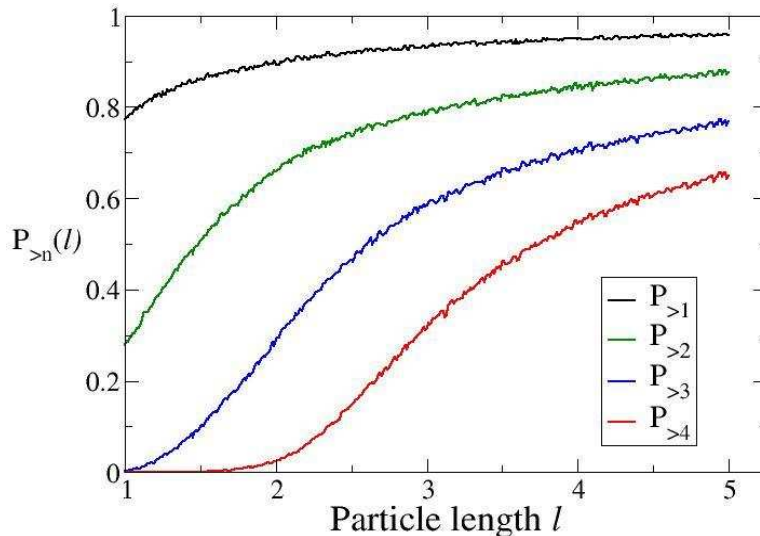
practical question of filtering actual particles, as they imply that even small rods will get trapped by very large holes, and long particles may pass through very small holes.

Another notable feature of Fig. 2 is that the peak probability to intersect  $n$  gridlines usually occurs at a length  $l$  less than  $n$ . Although counter-intuitive, this has a straightforward explanation. A rod whose left end is an infinitesimal distance  $\epsilon$  below and to the left of the origin immediately crosses two lines. A length only slightly larger than 1 can reach the next gridline. Because the rod makes the hypotenuse of a right triangle and intersects a gridline at every integer in  $x$  and  $y$ , and starts with two intersections at the origin, the relation between the number of intersections ( $n_x, n_y$ ) in the  $\hat{x}$  and  $\hat{y}$  directions is related to the length by the Pythagorean theorem such that  $n_x^2 + n_y^2 \leq l^2$  and the total number of intersections is  $2 + n_x + n_y$ . Assuming the equality, one finds the maximum number of intersections to be

$$N_{int}^{max} = 2 + \sqrt{2}l. \quad (10)$$

Figure 3 shows the theoretical maximum number of intersections from Eq. 10 as well as the maximum found in simulations(\*). The theoretical calculation assumes the rod to pass directly through the origin, an unlikely occurrence, the jump in the simulation occurs at a slightly larger length than predicted. Nevertheless, the close agreement between geometric theory and statistical simulation is encouraging.

In considering whether a rod gets filtered out by a mesh it seems more relevant to consider the cumulative intersection probability  $P_{>n}$ . This gives the probability that a rod intersects  $n$  or more grid lines. This is shown in Fig. 4. Each curve appears to asymptote to a slightly different value, and it is notable that  $P_{>2}$  does not asymptote



**Figure 4.** Cumulative intersection probability  $P_{>n}(l)$  giving the probability that a spherocylinder of diameter  $\sigma = 0.125$  and cylinder length  $l$  intersects  $n$  or more gridlines.

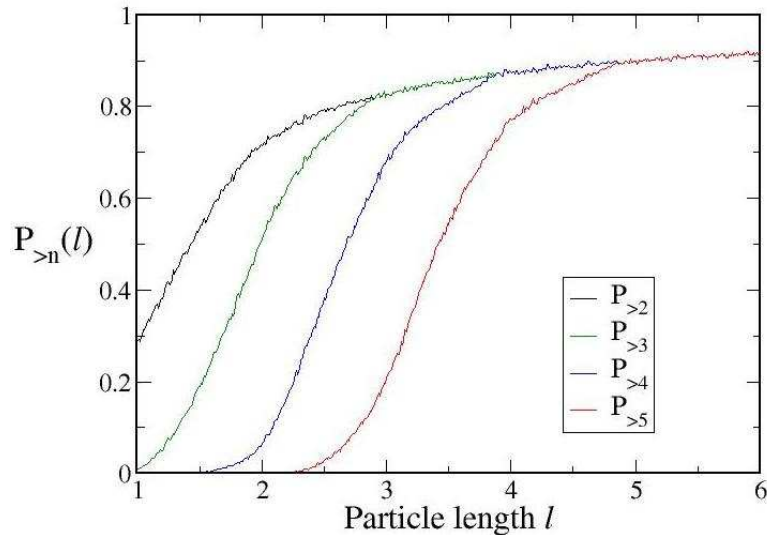
to 1, again implying that large rods may pass through a small hole.

#### 4. Rotations

The physics of how a particle moves after contacting the mesh is quite complicated, involving the dynamics of the surrounding fluid (which can be laminar or turbulent, depending on the rod velocity), lubrication between rod and mesh, as well as coupling between rods if the suspension density is large enough. As our final extension to the Buffon-Laplace problem we ignore all fluid-rod interactions and assume a frictionless interaction between rod and mesh. The only interaction is a normal force that causes a torque about the center of mass. The center of mass does not move in the plane of the mesh, only vertically, and so we consider rotations about this point. As the rod rotates the projected length onto the mesh changes, leading to the appearance or disappearance of contact points.

The dynamics of the projected length are determined by the center of mass height relative to the contact point. If the center of mass is below the contact point, then the torque causes the rod to rotate toward the vertical direction, reducing the projected length. Practically, we assume that the projected length shrinks until there are no more contacts with the mesh, at which point a rod would presumably flow unimpeded through the hole. If the rod's center of mass is above the contact point, then the torque causes the rod to rotate into the plane of the mesh, increasing the projected length to a maximum equal to the rod length. New contacts can arise and, if two points bracket the center of mass they can balance the gravitational torque and stabilize the final configuration.

We again calculate the cumulative intersection probabilities  $P_n(l)$ , now accounting for the rotations described above, and assume that  $N \geq 2$  is required for stability.



**Figure 5.** Cumulative intersection probability  $P_{>n}(l)$  after the rod has rotated due to the first mesh contact.

Figure 5 shows that these probabilities quickly asymptote to the same value as the particle length becomes large. This behavior should be contrasted with the slower convergence in Fig. 4. The explanation for this is straightforward. For any rod that rotates into the plane there is a minimum number of gridlines that must be crossed, and the asymptotic value of about  $P_{>n} \approx 0.9$  represents the probability that a long needle will rotate into the plane of the mesh. Thus, we conclude that approximately 10% of all particles will pass through a mesh even when the particle length is much larger than the mesh size. The probabilities shown in Figs. 4 and 5 are calculated with a finite particle width  $\sigma = 0.125$ , chosen so as to be comparable with the experiments described in the next section. As the particle width changes the asymptotic value of  $P_{>n}$  also changes.

We argue that the probabilities shown in Figs. 4 and 5 represent upper bounds on the probability that a particle is filtered by the mesh. We have observed in the simulations a rare, but not insignificant, number of final states with 2 or more intersections that do not seem plausibly stable. For example, cases exist where two contacts occur on the same side of the center of mass. These cannot balance the torque of gravity, and so should continue to rotate. For longer rods the second contact occasionally occurs when the lower side rotates up into the mesh, also a plausibly unstable configuration. While it is possible to adjust the Monte-Carlo simulations to remove these cases, we have not done so because we believe the impact is small, especially compared to the very real inertial and fluid dynamics effects we have also ignored. All of these would appear to cause destabilizing fluctuations in the particle motion, further reducing the number of particles that pass through. For example, rods sedimenting through a fluid tend to align in the direction of flow, and the distribution in zenith angle would have a vertical bias that decreases the intersection probability. Vibration

or fluid eddies would likely jostle particles on the mesh, which could only result in an increase in the number of particles that pass through. Thus, all modifications would decrease the intersection probability and the curves in Figs. 4 and 5 are likely the upper bound on the filtration efficiency.

## 5. Comparison with Experiment

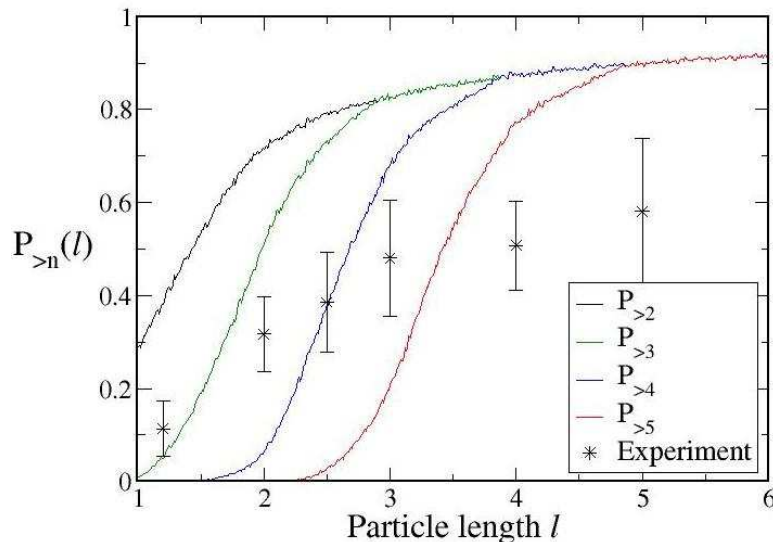
As a comparison, we have conducted experiments in which acrylic ( $\rho \approx 1.2 \text{ g/cm}^3$ ) cylinders of varying diameters  $D = 0.16 - 0.6 \text{ cm}$  and lengths  $L = 1 - 6 \text{ cm}$  pass through a mesh with square holes. The holes have a sidelength  $d = 0.6 - 1.3 \text{ cm}$ , resulting in a range of dimensionless lengths  $l \equiv L/d = 0.76 - 10$  and widths  $w \equiv D/d = 0.026 - 6$ . We report here data involving fixed dimensionless width  $w = 0.125$ , which were obtained from six separate combinations of  $L, D$ , and  $d$ . The dimensionless lengths ranged from 1-5.

Particles were dropped by hand in groups of  $N = 1 - 50$  particles and the number caught in the mesh recorded. Statistics were independent of  $N$  for  $N < 50$ . For  $N > 50$  the particles interacted with each other, resulting in statistically significant larger numbers of caught particles. To slow the particles' fall, thus reducing any inertial effects of bouncing on the mesh, they were released in a fluid (either water or water/glycerol mixture). The rods were released below the surface of the fluid to prevent the horizontal alignment that would occur on impact with the fluid surface. Rods sedimenting through a fluid will align in the direction of flow; to minimize this effect the mesh was placed 10 cm below the release point. We emphasize that these experiments were designed only to check the consistency of our probability calculations, since for the reasons described above we expect our calculations to overestimate the filtering probability.

Figure 6 confirms this behavior, with the experimental points appearing to asymptote at approximately 2/3 the value of the theoretical predictions. Whether this difference is due to particles orienting vertically as they fall through the fluid or any of the other physically plausible mechanisms described above is left for future experiments.

## 6. Conclusions

We have extended the classic Buffon-Laplace Needle calculation to account for finite width spherocylinders and orientations in three-dimensions. We find the probability for a particle to intersect the hole boundary  $P_{\text{int}}(l)$  increases with width, as expected. More importantly, the finite probability of vertical orientations results in  $P_{\text{int}}(l \gg 1) < 1$  indicating that even long rods may pass through a mesh. Similarly, the possibility of a rod spanning a corner leads to  $P_{\text{int}}(\ll 1) \neq 0$ , implying that even very short rods may get caught. We extended the analysis to include a probabilistic approach for particle rotations based on the location of the center of mass at the moment of first contact, and calculate cumulative probability distributions. Combined with the requirement of torque



**Figure 6.** Cumulative intersection probability  $P_{>n}(l)$  after the rod has rotated due to the first mesh contact.

balance, these geometric intersection probabilities can be compared to the probability that a rod might be caught by a mesh.

Although our approach ignores many physical principles, the probabilities calculated agree qualitatively with a simple experiment in which rods pass through a square-holed mesh. Both theoretical and experimental intersection probabilities increase monotonically from zero and asymptote below 1. The experimental asymptote is approximately two-thirds that of the theory, consistent with the idea that physical mechanisms that have been omitted in the theory (e.g. lubrication, inertia, and drag-induced alignment) would all tend to decrease the probability of being caught by the mesh.

That both theory and experiment asymptote below one even for very long rods encapsulates the fundamental difficulty of using a sieve or mesh to separate rods of varying length (but equal diameter). Very long rods will always have a finite, non-zero chance of orienting perpendicular to the plane of the mesh and thus passing through. Very short rods can still be caught by very large holes. The practical problem of rod filtration is therefore unlikely to be solved with a single mesh, and the analysis suggests that more complicated solutions are required.

## Acknowledgments

Maddie Pelz and Oscar Marcelino conducted the experiments to which our simulations were compared. This work was partially funded by the National Science Foundation DMR #0706353.

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